8.4 Gauss' Divergence Theorem

Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

Then

$$\operatorname{div} F = \nabla \cdot F$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where $F = (F_1, F_2, F_3)$

<u>Ex</u>: $F(x,y,z) = (x^2 y,\ z,\ xyz)$

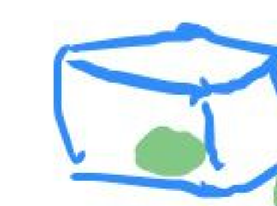
$$\Rightarrow \operatorname{div} F = \partial/\partial x\,(x^2 y) + \partial/\partial y\,(z) + \partial/\partial z\,(xyz)$$

$$= 2xy + 0 + xy = 3xy$$

physical meaning of divergence:

If F describes motion of gas

div $F(x,y,z)$ describes how a small volume unit at (x,y,z) changes in volume


(x,y,z)

If div $F(x,y,z) > 0 \Rightarrow$ expansion

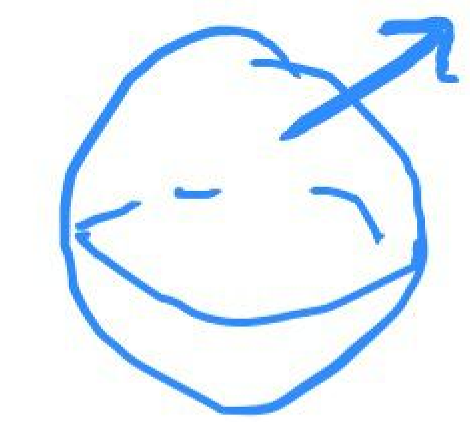
$< 0 \Rightarrow$ compression

Let W be 3-dim. region (say a ball, or a box)

∂W its boundary

Assume ∂W oriented with positive side = outside

$F: \mathbb{R}^3 \to \mathbb{R}^3$ vector field



Gauss Divergence Theorem:

Flow outside = (flux out)

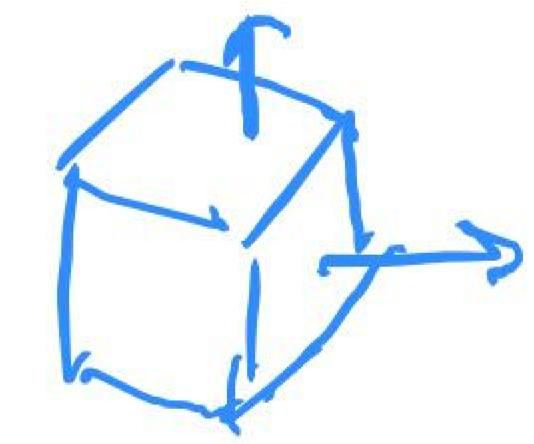
$$\boxed{\iint_{\partial W} F \cdot dS = \iiint_{W} \operatorname{div} F \, dV}$$

Examples

(1) W = unit box $[0,1] \times [0,1] \times [0,1]$

$$F(x,y,z) = (x^2y,\ z,\ 2xyz)$$

Calculate $\iint_{\partial W} F \cdot dS$, with pos. side = outside



Solution: • could do directly

but: involves parametrization of 6 faces of box

messy!

• use divergence theorem!

$$\operatorname{div} F = \partial/\partial x(x^2y) + \partial/\partial y(z) + \partial/\partial z(2xyz)$$

$$= 4xy$$

$$\Rightarrow \iint_{\partial W} F \cdot dS = \int_0^1\int_0^1\int_0^1 4xy\, dx\,dy\,dz = \int_0^1\int_0^1 2x^2y\Big|_0^1 dy\,dz = \int_0^1\int_0^1 2y\, dy\,dz =$$

$$= \int_0^1 y^2\Big|_0^1 dz = \int 1\, dz = 1$$

(2). Use Gauss divergence theorem to calculate $\iint_S F \cdot dS$

where $S = S_1 \cup S_2$ is the open can given by the two surfaces

S_1: $x^2+y^2=1$, $0 \le z \le 1$ (side of can)

S_2: $x^2+y^2 \le 1$, $z=0$ (bottom of can)

$F(x,y,z) = (xy^2, x^2y, z)$

<u>Solution</u>: W = cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$

$\Rightarrow \partial W = S \cup S_3$ ← top of can

$$\Rightarrow \boxed{\iint_S F \cdot dS = \iint_{\partial W} F \cdot dS - \iint_{S_3} F \cdot dS}$$

$$F(x,y,z) = (xy^2,\ x^2y,\ z)$$

$$\operatorname{div} F = y^2 + x^2 + 1$$

$$\Rightarrow \iint_{\partial W} F \cdot dS = \iiint_W y^2 + x^2 + 1 \, dV$$

$$= \int_0^1 \left(\iint_{x^2+y^2 \le 1} y^2 + x^2 + 1 \, dx\,dy \right) dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2+1)\, r\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left. \frac{r^4}{4} + \frac{r^2}{2} \right|_0^1 d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \frac{1}{4} + \frac{1}{2}\, d\theta\, dz = 2\pi \cdot \frac{3}{4} = \boxed{\frac{3\pi}{2}}$$

$S_3 = \text{top} = \{(x,y,1),\ x^2+y^2 \leq 1\}$

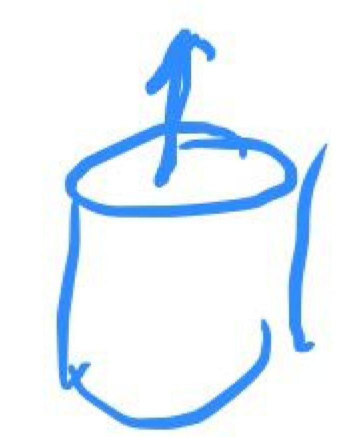
can parametrize as

$$\Phi(u,v) = (u,v,1)$$

check: $T_u = (1,0,0)$, $T_v = (0,1,0)$, $T_u \times T_v = (0,0,1)$

$T_u \times T_v$ points upwards

correct orientation!



$$\Rightarrow \iint_{S_3} F \cdot dS = \iint_{u^2+v^2 \leq 1} F(u,v,1) \cdot (0,0,1)\, du\, dv$$

$$= \iint_{u^2+v^2 \leq 1} (\sim, \sim, 1) \cdot (0,0,1)\, du\, dv$$

$$= \iint_{u^2+v^2 \leq 1} 1\, du\, dv = \pi$$

solution: $\iint_S F \cdot dS = \frac{3\pi}{2} - \pi = \boxed{\frac{\pi}{2}}$

Warning

Gauss divergence theorem may not hold if F has poles inside W

Important example: (relevant for Gauss' law for point charges)

$W =$ unit ball

$\vec{r}(x,y,z) = (x,y,z)$, $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$F(x,y,z) = \frac{\vec{r}}{r^3} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}^{\,3}}$$

⚠ $F(x,y,z) \longrightarrow \infty$ if $(x,y,z) \longrightarrow 0$

Claim: divergence theorem does not hold in this case.

Calculate div F!

$$\partial/\partial x \left(\frac{x}{\sqrt{x^2+y^2+z^2}^3} \right) = \frac{\sqrt{x^2+y^2+z^2}^3 \cdot 1 - \frac{3}{2}\sqrt{x^2+y^2+z^2} \cdot 2x}{(x^2+y^2+z^2)^3}$$

$$= \frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^3} \left(x^2+y^2+z^2 - 3x^2 \right)$$

sch.:

$$+ \partial/\partial y \left(\frac{y}{\sqrt{}^3} \right) = \quad '' \quad \left(x^2+y^2+z^2 - 3y^2 \right)$$

$$+ \partial/\partial z \left(\frac{z}{\sqrt{}^3} \right) = \quad '' \quad \left(x^2+y^2+z^2 - 3z^2 \right)$$

$$\text{div } F = \quad '' \quad \left(3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2 \right) = 0$$

$\Rightarrow$ $\operatorname{div} F = 0$ for all $(x,y,z) \neq 0$

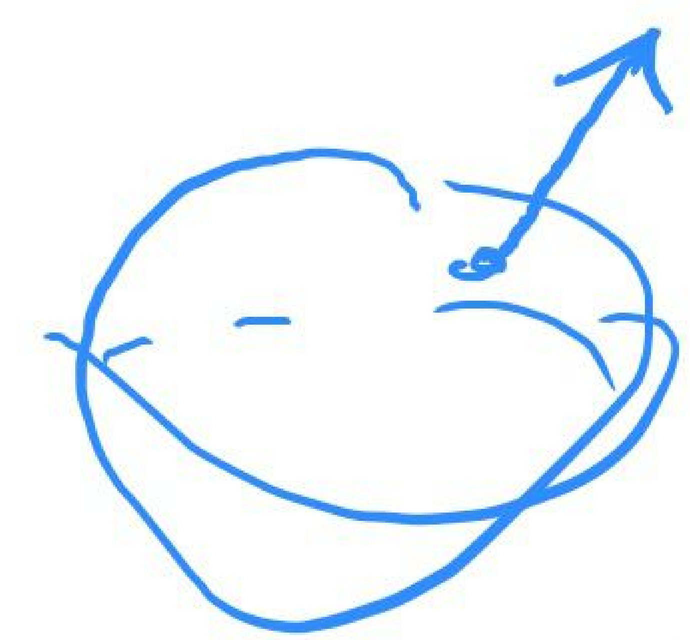
$\Rightarrow$ $\iiint_W \operatorname{div} F \, dV = 0$

seen:
in our case
$\vec{n}(x,y,z) = (x,y,z)$
$F(x,y,z) = (x,y,z)$

let's calculate $\iint_S F \cdot dS$

on S we have $x^2+y^2+z^2=1$

$\Rightarrow$ $F(x,y,z) = (x,y,z)$ $\underline{\underline{\text{on } S}}$



$\vec{n}(x,y,z) = (x,y,z)$

of sphere of radius 1

one can show:

$$\boxed{\iint_S F \cdot dS = \iint_S (F \cdot \vec{n}) \, dS}$$

surface integral for vector fields

surface integral for functions

$\vec{n}$ = normal vector of length 1

$$\Rightarrow \iint_S (F \cdot n)\, dS = \iint_S (x,y,z)\cdot(x,y,z)\, dS$$

$$= \iint_S 1\, dS$$

$= $ area of sphere

$$= 4\pi \neq 0 = \iiint_W \operatorname{div} F \cdot dS$$